Creative Thinking Solutions

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1 Kelsie's Coin Collections

a) Since a collection must have at least one penny and at least one nickel, think about the other coins in the collection. Let's say there are $c + 2$ coins in the collection (the +2 is for the penny and nickel we must have). Then aside from the coins Kelsie must have, there are c other coins in the collection. To make the value as large as possible, these should all be nickels, giving Kelsie a total of 1 penny and $c + 1$ nickels, totaling $5c + 6$ cents. To make the collection have as small a value as possible, take the extra coins to all be pennies, making a total of $c + 6$ cents. The difference is $4c$, which must be a multiple of 7. The smallest c can be so that $4c$ is a multiple of 7 is $c = 7$. There can be either 8 nickels and a penny (41 cents) or 8 pennies and a nickel (13 cents)—the difference is 28 cents which is indeed a multiple of 7. Since $c = 7$, the number of coins in the collection is $c + 2 = \boxed{9}$.

b) Since there are two nickels for every penny, each penny is part of a group of coins totaling 11 cents. So the total value of the collection is a multiple of 11. We continue the Fibonacci sequence until we meet a multiple of 11: 1, 1, 2, 3, 5, 8, 13, 21, 34, **55**. Now $55 = 5 \times 11$, so there are five of the penny-nickel-nickel groups, and $5 \times 3 = \boxed{15}$ coins.

2 Kelsie's Colored Coins and Connecting Curves

a) Each red coin is connected to each blue coin, so that means there are $4 \times 3 = 12$ curves connecting red and blue coins. There are similarly 4×3 curves connecting red and green coins. Finally, there are $3 \times 3 = 9$ curves connecting blue and green coins. That makes a total of $12 + 12 + 9 = \boxed{33}$ curves

b) If Kelsie draws out r red coins, b blue coins, and g green coins, by analogy with the example and previous problem she will need to draw $rb + rg + bg$ curves. We seek to make this as large as possible. First, for any particular number r of red coins, we could re-write the expression for the number of curves as $r(b + g) + bg = r(20 - r) + bg$ (because there are 20 coins in total, $r + b + g = 20$). Now once r is chosen, to make bg as large as possible with the remaining coins, there should be an equal number of blue and green coins (or as close to equal as possible, if there aren't an even number of blue and green coins together). So the maximum number of curves occurs when $b = g$ (or they're different by one). There's nothing special about choosing r; if we had chosen the number of blue coins, the maximum number of lines would be achieved when $r = g$ (or as close as possible) and similarly if g is chosen we need $r = b$. The closest we can come to making $r = b = g$ is to have two of them equal to 7, and the third equaling 6. That would total $7 \cdot 7 + 7 \cdot 6 + 7 \cdot 6 = \boxed{133}$ curves.

3 Roy's Restricted Robots

The problem can be solved by playing around with the numbers, "brute force" style, and the answers shown below will do it that way. But first, a more mathematical explanation.

If the robot's code is (f, b), find the number g which is the greatest common divisor of f and b. Claim: the robot can move any multiple of g spaces either forward or backward, but may not visit any other locations. Here's why.

If g is the greatest common divisor of f and b, then f and b are both multiples of g. That is $f = sg$ and $b = tg$. After m forward moves and n backward moves (since these are just adding or subtracting, it doesn't matter what order they're in) the robot has moved $mf - nb$ spaces forward (backward, if the number is negative). But this is equal to $msg + ntg = (ms + nt)g$, a multiple of g. So nothing other than multiples of g are achievable.

Next, if the robot can move a spaces ahead, for any a, it can also move a spaces backward. For let $a = mf - nb$ so the robot gets to a by moving forward m times and backward n times. Notice that $-a = -mf + nb$. So $-a$ is achieved by moving forward $-m$ times and backward $-n$ times. Of course, that's not allowed! But by moving forward b times and backward f times, the robot returns to its starting spot. Choosing some large number k of times to do this, then move ahead $kb - m$ times and backward $kf - n$ times and both of those numbers will be nonnegative, so those moves will get the robot to $-a$.

Finally, we know that every move puts the robot some multiple of g away from its starting space. Let's say the smallest possible multiple of g that can be achieved is kg. If k is not equal to one, then kg is either not a divisor of f or it is not a divisor of b, because g is the *largest* common divisor and kg would be larger. For the sake of argument, say that kg is not a divisor of b. Then by moving ahead kg spaces until the robot is more than b away from its starting point, it can now move b backward, and be *closer* to its starting point than kg. But kg was supposed to be the smallest multiple of g the robot could achieve. So k must equal one and the robot can get g spaces away.

In case kg is not a divisor of b, do the same process using $-kg$ (which is achievable from the previous paragraph) and reversing the roles of f and b.

The foregoing is (an admittedly awkward) description of *Bezout's Theorem* and its proof: that for any numbers f and b, the greatest common divisor of f and b is the smallest possible positive integer that can be written in the form $mf - nb$.

a) Notice that the greatest common divisor of 18 and 14 is 2. By moving $4 \times 18 = 72$ forward and then $5 \times 14 = 70$ back, the first robot can move to 2. If it can move 2, it can move any multiple of 2. And it can't move any odd number, because both its forward and backward moves are even numbers, so once it's on an even number (it starts at zero!) it stays on them. The smallest move the second robot can make would be four forward and one backward movements, or $4 \cdot 9 - 1 \cdot 33 = 3$. (Again, notice that the greatest common divisor of 9 and 33 is 3.) Since both moves are multiples of 3 no other numbers are achievable. So the first place these robots can meet up must be a multiple of 2 and of 3, and the smallest common multiple of 2 and 3 is $\boxed{6}$.

b) The first robot can move forward four times then backward seven times to get to $4 \cdot 27 - 7 \cdot 15 = 108 - 105 = 3$. So this robot can move to any multiple of 3. The second robot moves to multiples of $3 \cdot 35 - 2 \cdot 49 = 105 - 98 = 7$. But starting at 1, it can then move to 8, 15, 22, 29, ... The first of these numbers that is divisible by 3 is $\boxed{15}$.

4 Andrew's Awesome Algorithm

a) Notice that while there is a positive power on both the 2 and the 3, multiplying by $\frac{1}{6}$ drains one away from each power. Since $b \leq a$, the power on the 3 will drain away first, after multiplying by $\frac{1}{6}$ b times. This will leave 2^{a-b}. Now the number will keep getting multiplied by $\frac{5}{2}$, moving powers from the 2 onto the 5. So when the process is complete, the final number will be 5^{a-b}. Thus, the power on the 5 is $\boxed{a-b}$.

b) The first part of this problem gave the desired answer if $a \geq b$. But if $b \geq a$ then after mulitplying by $\frac{1}{6}$ as many times as possible, the power on the 2 has been drained away, leaving 3^{b-a}. So simply append the fraction $\frac{7}{3}$ to move this exponent onto the 7. Thus, the final list is $\boxed{\dfrac{1}{6}, \dfrac{5}{2}, \dfrac{7}{3}}$. The final two fractions could be swapped: $\dfrac{1}{6}, \dfrac{7}{3}, \dfrac{5}{2}$.

An *algorithm* is a specific sequence of steps that is designed to accomplish some set task. In this problem, the list of fractions is kind of like a computer program, and the number selects which instruction to execute next. When no more steps can be executed, the program has finished.

This is a very brief look at the FRACTRAN programming language invented by the late John H. Conway. Lists of fractions like these can actually be used to compute anything that any computer can compute, albeit very slowly. For instance, the following program takes $2^a \cdot 3^b$ and outputs 5^{ab}:

$$\frac{455}{33}, \frac{11}{13}, \frac{1}{11}, \frac{3}{7}, \frac{11}{2}, \frac{1}{3}.$$