8th Grade Team Solutions

IMSA *Mu Alpha Theta*

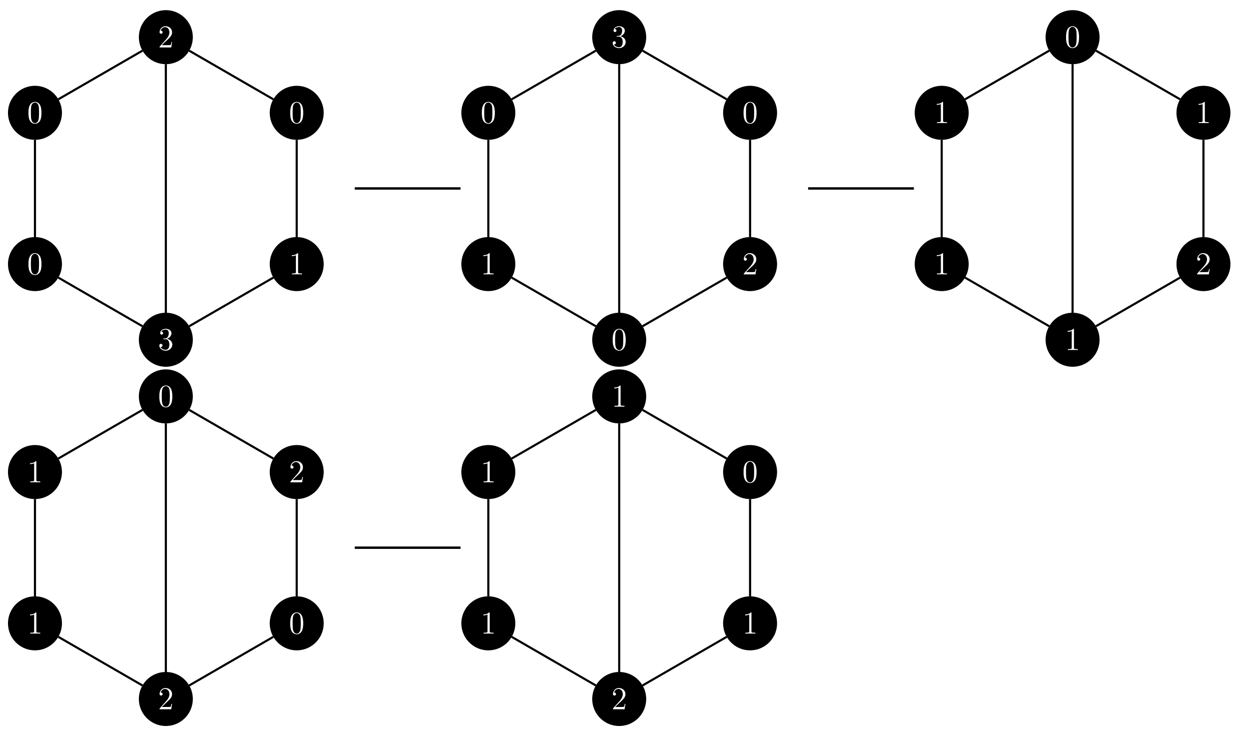
March 11, 2026

1. $2025^2+2023^2-2024^2-2022^2? = (2025^2-2024^2)+(2023^2-2022^2) = (2025+2024)(2025-2024)+(2023+2022)(2023-2022) = 4049+4045 = \boxed{8094}$.

2. The ages of the people don't matter. The average age is $\frac{300}{6} = \boxed{50}$

3. Notice that these numbers ara in the form $(a+b, a^2+b^2) whichequals (a+b, (a+b)^2-2ab)$. When trying to find the gcd, this is equivalent to $(a+b, 2ab)$. From here, it's much more obvious that the gcd is 1, since the prime factors of $2(100)(123)$ are $2, 5, 3, 41$ and none of these go into 223.

4. This should be straightforward, just a fun computational puzzle!:



   We count $\boxed{4}$ moves.

5. Since the time traveler is from 1428 but has gone back in time, and we know the year starts with the number 3, we can determine that the year is somewhere in the 300s (since the next largest year beginning with 3 is 3000, which is in the future). Since the

hundreds digit is a 3, we know the sum of the remaining 2 digits is 9, based on what the second local tells her. Since the number is divisible by 5 but not by 10, it must end in 5, as any number divisible by 5 ends in either 5 or 0, but if it ended in 0 it would be divisible by 10. Therefore the year is $\boxed{345}$.

6. Given this, we can always interpret his next score as $\frac{T_n}{4}$, and $T_{n+1} = \frac{T_n}{2} + T_n = \frac{3}{2} * T_n$. Therefore, since his first three test scores add up to 192, we can just think of this process as $\frac{\frac{3}{2}^{n-3}}{n} \cdot 192 > 90$. Simply following this function for 3 iterations, we get an average of 108 after $\boxed{6}$ total tests.

7. Notice that Andrew's chessboard isn't unique and easy to replicate. 0, empty squares is the minimum. It can be done by taking 8 of the 13 rooks, and then aligning them such that there is a rook present in every row and column. This can be guaranteed by simply placing it along the diagonal. For Roy's chessboard, since we've already placed 8 rooks, there are $13 - 8 = 5$ rooks left to place! And now to do this, we can *copy* the same technique for Andrew's board with our smaller diagonal of 5. This essentially would take the 8×8 chessboard, and cross out 5 rows and columns each, leaving an area of 3×3 **safe zone** where we have no attacks received by any of the placed rooks. This becomes our optimal setup, as for Andrew's board we cannot place another rook without being in attack of another rook already placed thus this is the optimal setup. From here, note that $R = 9$ and $A = 0$, thus the result $R - A = \boxed{9}$.

8. We have $0.97x = 0.96y$ if x, y are to represent the sizes of the class of 1997 and 1996, respectively. This yields $97x = 96y$. Since 96 and 97 have no common factors other than 1, this implies that y must be divisible by 97 and x must be divisible by 96 (since both sides of the equation must be integers, because they represent people). This implies that $x = 96k$ for some integer k. Then we have $y = \frac{97 \cdot 96k}{96} = 97k$. Since 97% of x is 96% of y, $0.97x = 0.97 \cdot 96k = 93.12k$, which is also an integer. Clearly, to minimize the value of $x + y$ we must find the least value of k such that $93.12k$ is an integer. $\frac{9312k}{100} = \frac{2328k}{25}$, so $k = 25$. Then, $x + y = 96k + 97k = 193 \cdot 25 = 192 \cdot 25 + 25 = 48 \cdot 4 \cdot 25 + 25 = 4800 + 25 = \boxed{4825}$.

9. Let x be the first term. The sum of the 2025 integers is $2025x + (1 + 2 + \cdots + 2024) = 2025 \cdot 2026$. The sum from 1 to 2024 can be described as $\frac{2024}{2} = 1012$ pairs of sums that each add up to 2025 (e.g. $1 + 2024, 2 + 2023, 3 + 2022$, etc.). So, we have $2025x + 1012 \cdot 2025 = 2025 \cdot 2026$. Dividing by 2025 on both sides, we have $x + 1012 = 2026 \implies x = 1014$. If the first term is 1014, the last term is $1014 + 2024 = 3038$, so the 2024th and 2023rd terms are 3037 and 3036, so we have $3037 + 3036 = \boxed{6073}$.

10. Pieces of candy come in masses of 3g, 7g, and 11g. Sonit is making a bag of candy and he wants the overall average of the bag to be exactly 8g per piece. What is the minimum number of pieces he needs?

11. First we let $u = x^2$. Then, the equation becomes a quadratic: $4u^2 - 61u + 225$. We can factor this quadratic using any number of methods. For instance, using master product, we want to find two numbers that sum to -61 and multiply to $4 \cdot 225 = 900$. Since $\sqrt{900} =$

30 and $-30-30=-60$ is pretty close to -61, we can guess that the two numbers will be around -30. Then, the two numbers have to be -36 and -25. So, our quadratic becomes $4u^2-25u-36u-225 = u(4u-25)-9(4u-25) = (u-9)(4u-25) = (x^2-9)(4x^2-25)$. Using difference of squares we can factor this as $(x-3)(x+3)(2x-5)(2x+5)$. Plugging in $x=37$, we have $(34)(40)(69)(79)$. Prime factorizing, we have $(2\cdot 17)(2^3\cdot 5)(3\cdot 13)(79)$, so the distinct prime factors of this number are 2, 17, 5, 3, 13, and 79. Note that you can quickly check that 79 is prime by checking divisibility by prime factors up to $\sqrt{79}\approx 9$. The sum of the distinct prime factors is $2+17+5+3+13+79=\boxed{119}$.

12. The ball can go left or right at each peg, so there are $2^3=8$ possible paths for each ball. The paths that lead to box 3 are:

- LRR
- RRL
- RLL

Thus, there are 3 paths that lead to box 3. The probability that one ball lands in box 3 is $\frac{3}{8}$. Since the balls are independent, the probability that both balls land in box 3 is:

$$P(\text{both in box 3}) = P(\text{ball 1 in box 3})\cdot P(\text{ball 2 in box 3}) = \left(\frac{3}{8}\right)^2 = \frac{9}{64}.$$

Thus, the probability that both balls land in box 3 is $\boxed{\frac{9}{64}}$.

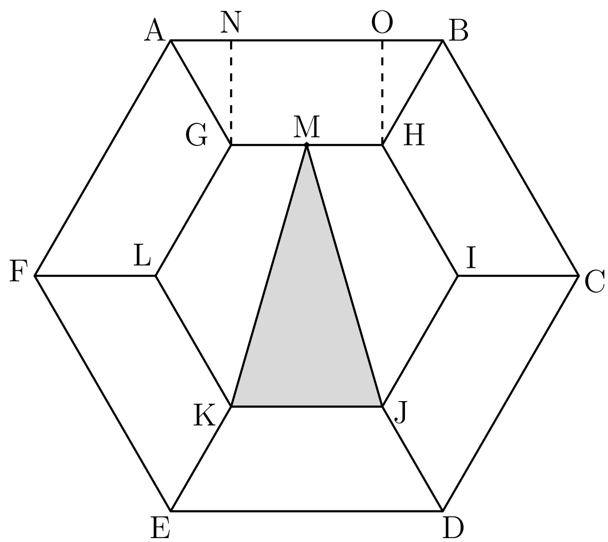
13. Using Pythagorean theorem, we find that $\mathrm{AC}^2 + \mathrm{AB}^2 = \mathrm{CB}^2$. This is also equal to $(\mathrm{CD} + \mathrm{DB})^2$. Expanding the latter gives us $\mathrm{CD}^2 + \mathrm{DB}^2 + 2\mathrm{CD}\cdot\mathrm{DB}$. We also have that $\mathrm{AD}^2 + \mathrm{DB}^2 = \mathrm{AB}^2$, and that $\mathrm{AC}^2 = \mathrm{AD}^2 + \mathrm{CD}^2$. We can substitute the latter two equations into the first one to get that $\mathrm{AD}^2 + \mathrm{DB}^2 + \mathrm{AD}^2 + \mathrm{CD}^2 = \mathrm{CD}^2 + \mathrm{DB}^2 + 2\mathrm{CD}\cdot\mathrm{DB}$. This results in $2\mathrm{AD}^2 = 2\mathrm{CD}\cdot\mathrm{DB}$, or $\mathrm{AD}^2 = \mathrm{CD}\cdot\mathrm{DB} \implies \mathrm{AD} = \sqrt{14}$. The total area is: $\frac{1}{2}\cdot 14\cdot\sqrt{14} = \boxed{7\sqrt{14}}$.

14. Note that $g(h(x)) = h(x) = 1$, so we have $x^3+3x^2-10x-23=1$, or $x^3+3x^2-10x-24=0$. We are also given that one of the roots is -4, which means that $(x+4)$ is a factor of this expression. We can consider the equation $(x-a)(x-b)(x+4) = x^3+3x^2-10x-24$, where a and b are the other two values for x. The constant term -24 is derived by the constant terms in each of the factors of the expression, since it does not contain x. In other words, $-24 = -a\cdot -b\cdot 4$, which implies that $a\cdot b=-6$. Also, consider that when you foil a factored expression, each term is generated by choosing a term from each factor (e.g. x or -4), and multiplying all of them together. With this idea, we can quickly find the coefficient of the x terms by choosing one of the x terms across the three factors, and choosing the constant terms for the other two factors. Choose $x, -b, 4$, $-a, x, 4$, $-a, -b, x$, to get the coefficient of x as $-4b-4a+ab=-10$. We substitute $ab=-6$ to get $-4b-4a=-4$, $\implies b+a=1$. We can see that one of the two terms must be 3, and the other must be -2. Since the problem only wants the positive one, our answer is $\boxed{3}$.

15. Factor $x^{10} \cdot x^{100}$ from the expression to get $x^{10} \cdot x^{100}(2 + 3x + 5x^{91} + 7x^{90})(11 + 9x + 13x^{991} + 17x^{990})$. Multiplying by x^{110} does not change which coefficients are even, so we focus on $(2 + 3x + 5x^{91} + 7x^{90})(11 + 9x + 13x^{991} + 17x^{990})$. Note that each term in the expansion comes from multiplying one term from the first parentheses with one term from the second. Then, we can consider two cases. Case 1: Products involving the coefficient 2. Since 2 is the only even coefficient, it is the only term in the parentheses that can contribute an even coefficient in the final expansion on its own. The degrees of the permutations of pairs of terms chosen from the two sets of parentheses where the first term is 2 are 0, 1, 991, and 990, but some of these also receive contributions from the products of odd terms ($3x \cdot 11$ and $3x \cdot 17x^{990}$ contribute to the terms with degree 1 and 991, respectively). Accounting for this overlap, two terms remain even. Case 2: Products formed by terms with only odd coefficients. As we have seen previously, sometimes terms in the final expansion can be contributed two by multiple pairs of terms from the original factored parentheses. Therefore, it is possible for two such pairs to contribute to the same final term, resulting in an even sum, even if neither of the terms had an even coefficient to begin with. Note that $5x^{91} \cdot 11$ and $7x^{90} \cdot 9x$ both contribute to the same term, and $5x^{91} \cdot 17x^{990}$ and $7x^{90} \cdot 13x^{991}$ also both contribute to the same term. There are only two such cases, giving two more terms in the final expansion that have an even coefficient. Adding both cases, there are $\boxed{4}$ terms with even coefficients in the full expansion.

16. Let's see what we can find with our general form of the terms in this series, $n \cdot n!$. We can rewrite this as $n(n!) = n!(n+1-1) = n!(n+1) - n!$. By the definition of a factorial, $n!(n+1) = (n+1)!$, so we continue to rewrite $n!(n+1) - n! = (n+1)! - n!$. We finally found a convenient form to see a pattern! Let's see what our new form does to the series: $(1 \cdot 1!) + (2 \cdot 2!) + (3 \cdot 3!) \ldots (41 \cdot 41!) = (2! - 1!) + (3! - 2!) + (4! - 3!) \cdots + (41! - 40!) + (42! - 41!)$. Notice that all the terms in between the first and last term cancel out, leaving us with $-(1! + 42!) = 42! - 1 \Rightarrow k = 42, m = 1$. Our answer is $k - m = 42 - 1 = \boxed{41}$.

17. First, note that all the base angles of all the trapezoids are congruent, and add up to the sum of the interior angles of the outer hexagon. The sum of the interior angles of a polygon with n sides is $180(n-2) = 180(4) = 720°$, which makes $\angle GAB = \frac{720}{12} = 60°$. Drawing a line from vertex G to AB, we form a 30-60-90 triangle.



Given $\overline{GA} = 2$, using 30-60-90 triangle properties, we find that $\overline{AN} = 1$. Note that by symmetry we can draw a similar triangle on the other side of trapezoid $ABHG$, and that $\overline{AB} = \overline{AN} + \overline{OB} + \overline{NO}$. Since $NOHG$ forms a rectangle, $\overline{NO} = \overline{GH} = \overline{AB} - \overline{AN} - \overline{OB} = 6 - 1 - 1 = 4$, and hence $\overline{KJ} = 4$. To find the height of ΔKMJ, split hexagon $GHIJKL$ into six congruent equilateral triangles of side length 4. Note that the height of ΔKMJ would have the same height as two of these equilateral triangles. The height of an equilateral triangle of side length 4 is $2\sqrt{3}$, which makes the height of ΔKMJ $4\sqrt{3}$. $\frac{4\sqrt{3}\cdot 4}{2} = \boxed{8\sqrt{3}}$.

18. Let r be the radius of the cylinder that Andrew cuts out of the donut. Now, let's consider the sphere that we have to cut out. When the diameter of this cylinder $(2r)$ is smaller than 2, the height of the original donut, then we have to choose r as the radius of our maximal sphere. Similarly, if our diameter is larger than 2, then we have to choose 1 as our radius as that is the largest sphere that can fit in the cylinder. So, any $r \geq 1$ caps the sphere volume at $\frac{4}{3}\pi$. Additionally, for the sphere to have volume $\frac{9}{16}\pi$, $\frac{4}{3}r^3\pi = \frac{9}{16}\pi \Rightarrow r^3 = \frac{3^3}{4^3} \Rightarrow r = \frac{3}{4}$. So, our interval of r is $[\frac{3}{4}, \frac{3}{2}]$, so the probability that Andrew chooses a sufficient r is $\boxed{\dfrac{1}{2}}$.

19. Notice that the way that the lines are drawn is symmetric to the diagonal $\overline{DB}$, meaning that the same trisections and bisections are applied from opposite vertices. Due to this, we can see that the division of ΔCBD and ΔADB is mirrored, and since we only care about ratios, we can analyze just one of the triangles: the ratios found there will hold for the other triangle as well.

    Now, the problem becomes simpler. We want to compare the areas of ΔEHC and ΔHBC. We can see that the two triangles share the same height, since their bases are collinear and they converge to the same vertex. Now, we can simply compare the lengths of their bases! Suppose without loss of generality that $\overline{DB}$ has length 9, then $\overline{EF}$ has length $9 \div 3 \div 2 = 1.5$, and $\overline{FG}$ has length $9 \div 3 \div 3 = 1$ (if you're confused, see the instructions on how $\overline{DB}$ is dissected). We see that $\overline{EH} = 1.5 + 1 = 2.5$, and we similarly see that $\overline{HB} = 1 + 3 = 4$. Hence, the ratio is $\frac{2.5}{4} = \boxed{\frac{5}{8}}$.

20. Consider a function g, where $g(x) = f(x) - x$, then $g(1) = f(1) - 1 = 0$, a root of the polynomial g, and so on subsequently for $x \in \{1, 2, 3, 4\}$. Thus, these must **all** be roots of g, so $g(x) = (x-1)(x-2)(x-3)(x-4)$, and therefore $f(x) = x + g(x) = x + (x-1)(x-2)(x-3)(x-4)$. Computing $f(5) = 5 + (5-1)(5-2)(5-3)(5-4) = 5 + 4 \cdot 3 \cdot 2 \cdot 1 = 5 + 24 = 29$. Therefore $\boxed{29}$.