Creative Thinking

IMSA *Mu Alpha Theta*

March 5, 2025

1 Fogel's Fudge

As a reward for their performance on the Junior High Math Contest, Coach Fogel has just made a large batch of fudge for his math team.

a) For the individual competitors, he was going to give the first-place finisher 1 pound of fudge, the second-place finisher $1/2$ pound, the third-place finisher $1/3$-pound, and so on down to the 24th-place finisher receiving $1/24$ pound. But on the way to math team practice he got hungry. He ate $1/2$ pound of the first-place finisher's fudge, $1/3$ pound from the second-place finisher, $1/4$ pound from the third-place finisher, all the way on down to eating $1/25$ pound of the 24th-place finisher's fudge. When he finally gave the fudge to the actual students, how much fudge did they receive in total?

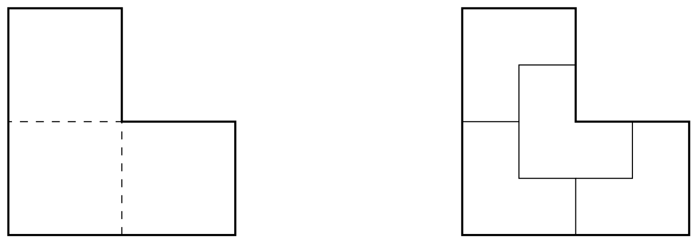
b) For the team competitors, Coach Fogel made another large batch of fudge. He decided to give the fudge to the teams according to these rules:

   - To the first-place team, he gave 1 pound plus $1/10$ of the fudge remaining (after the pound has been taken) in the batch.
   - Then, to the second-place team, he gave 2 pounds plus $1/10$ of the remaining (after the first team and the second team's 2 pounds) fudge.
   - Then, to the third-place team, he gave 3 pounds plus $1/10$ of the remaining fudge.
   - ...
   - Finally, to the team that finished in last place, he gave n-pounds plus $1/10$ of the remaining fudge.

   Amazingly, Coach Fogel gave away *all* the fudge in the batch, and *every* team received exactly the same amount of fudge. How much fudge was in the batch?

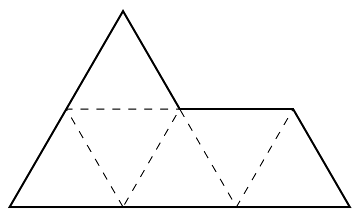
2 Roy's Rep-tiles

A geometric shape is called a *rep-tile* if it can be cut into smaller pieces that are exactly similar to the original shape. That is, if you use the shape as a "tile" you can put them together to make a larger copy of the original shape. For instance a shape consisting of three squares glued together into an L-shape is a rep-tile as shown below:

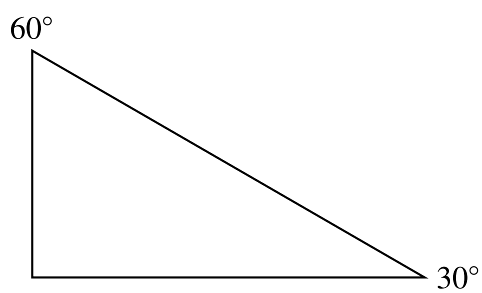


If the rep-tile uses n copies of its smaller self, it is called rep-n. The L-shape above is rep-4.

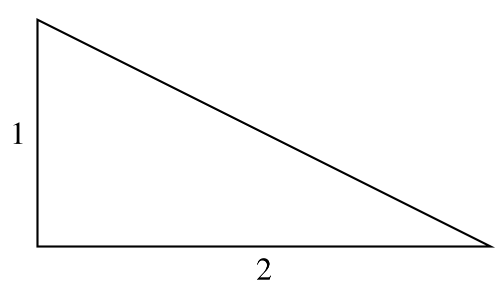
a) The shape below, called the sphinx tile, is comprised of 6 equilateral triangles. Create a drawing showing that it is a rep-4 rep-tile. (Make sure to transfer your drawing to the answer sheet!)



b) Draw a picture to demonstrate that the 30-60-90 triangle is a rep-3 rep-tile.



c) Draw a picture to demonstrate that the right triangle with legs of length 1 and 2 is a rep-5 rep-tile.



3 Max's Continued Fractions

Max enjoys putting fractions inside other fractions (inside another fraction inside another ...) the result is called a *continued fraction*. An example might be

$$4 + \cfrac{1}{2 + \frac{1}{3 + \frac{1}{6}}}$$

which is notated $[4, 2, 3, 6]$ for short. It can be simplified to a regular fraction by the usual arithmetic:

$$4 + \cfrac{1}{2 + \frac{1}{3 + \frac{1}{6}}} = 4 + \cfrac{1}{2 + \frac{1}{19/6}} = 4 + \cfrac{1}{2 + \frac{6}{19}} = 4 + \frac{1}{44/19} = 4 + \frac{19}{44} = \frac{195}{44}$$

Max actually remembers a similar example showing up on the 8th-grade Team Contest, though all the numerators there were 5's and not 1's, but it was infinite—the fractions inside of fractions kept going forever. Max tried to figure out $[3, 3, 3, 3, \ldots] = 3 + \cfrac{1}{3 + \frac{1}{3 + \frac{1}{3 + \cdots}}}$. To start, he noticed that the whole fractions was actually inside itself, so he set the whole fraction equal to x and then was able to write

$$x = 3 + \cfrac{1}{3 + \frac{1}{3 + \frac{1}{3 + \cdots}}} = 3 + \frac{1}{x}.$$

Multiplying both sides by x he found $x^2 = 3x + 1$ or $x^2 - 3x - 1 = 0$. The quadratic formula then showed that $x = \frac{1}{2}(3 \pm \sqrt{13})$ and Max kept only the plus sign, since the minus sign gave him a negative answer which doesn't make sense because obviously x is bigger than 3!

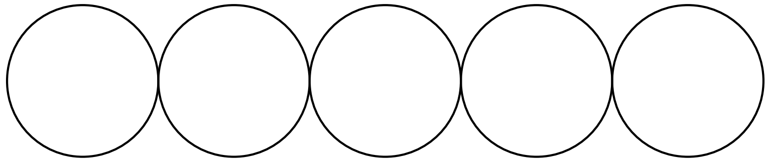
Help Max out:

a) Find the $[a, b, c, d, e, \ldots]$ notation for $1247/372$. (Hint: find the whole part!)

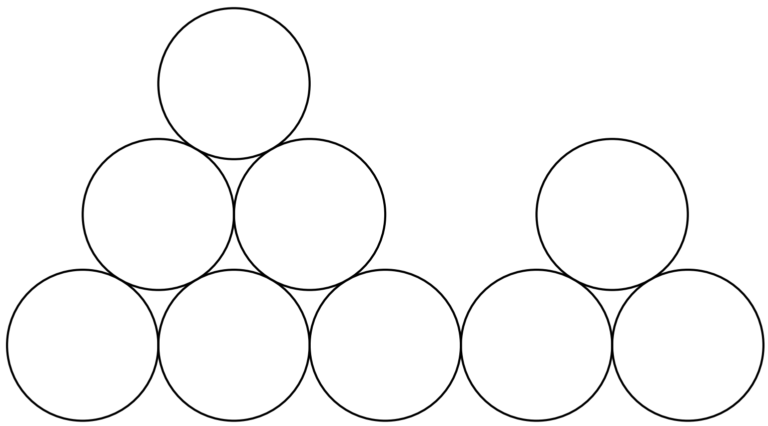
b) Find the value of $[1, 2, 1, 2, 1, 2, \ldots]$ where it repeats forever.

4 Claire's Casks

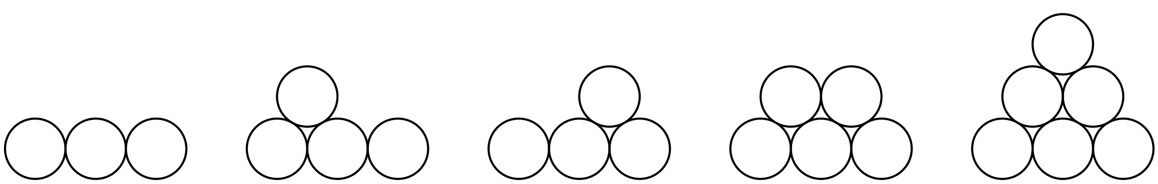
Claire has a number of barrels to store in her basement. They seem to fit best when lying on their sides, and the basement room is just wide enough to lay five barrels side-by-side:



She can now stack more barrels on top of this row. She has to keep the five barrels in the bottom row or they may roll around causing the barrels on top to fall and break, but higher rows do not have to be complete, but barrels that aren't supported by two directly beneath it will fall and break. One legal way to stack the barrels is shown below:



If Claire's basement had only been wide enough for three barrels, there would be five legal stacks, as shown below:



Had the basement been wide enough for four barrels, there are 14 legal stacks.

a) Since Claire has room for a five-barrel base, how many legal stacks are there? (Remember, the bottom row must have all five barrels in it.)

b) If there are at least three barrels in Claire's stack that are not on the bottom row, how many possible stacks are there?