8^{th} Grade Team Contest Solutions

IMSA *Mu Alpha Theta*

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1. Note that $1/(1/2) = 2$, so as the fractions get deeper and deeper they alternate in value between $1/2$ and 2. Since there is a 2 six fractions deep, it alternates back and forth three times and ends at $\boxed{2}$.

2. Factoring into primes, $75 = 3 \cdot 5^2$. A cube has all of its prime factors to powers that are multiples of three, so 75 needs to be multiplied by $3^2 \cdot 5 = \boxed{45}$ to become a cube.

3. At first, the numbers on the board sum to $1 + 2 + 3 + \cdot + 32 = 528$ (this can be found by adding or using the formula $1 + 2 + \cdots + n = n(n + 1)/2$). Each student who goes to the board causes the total sum of the numbers on the board to decrease by 1. Since 31 students come to the board the final number left written on the board will be $528 - 31 = \boxed{497}$.

4. Let $a < \sqrt{n} \leq a + 1$. We need to look for numbers n that are multiple of the least common multiple of the number from 1 to a. First, if $a = 5$ then the least common multiple of 1, 2, 3, 4, and 5 is 60, so number less than 60 is a multiple of all of these. Unfortunately, if $a = 5$ then $n \leq (5 + 1)^2 = 36$. So a is at most 4. If $n = 4$ then the least common multiple of 1, 2, 3, and 4 is 12. We need $n \leq (4 + 1)^2 = 25$. The largest that is at most 25 and is divisible by 12 is $\boxed{24}$.

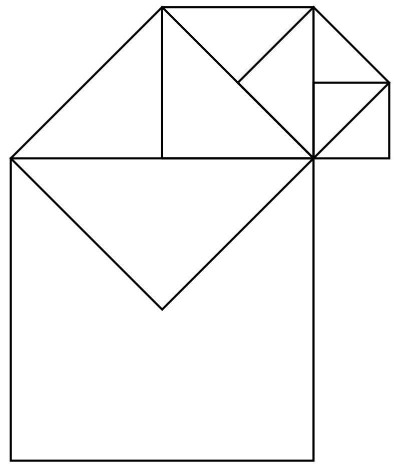
5. If ab is a two-digit number the product of its digits is either a one- or two-digit number. So the possible cubes that could be $a \cdot b$ are 1, 8, 27, and 64.

   - To get $a \cdot b = 1$ we must have $a = b = 1$. Then $a + b = 2$ is prime, so $ab = 11$ is one such number.
   - To get $a \cdot b = 8$ we must have $1 \cdot 8$ or $2 \cdot 4$. But neither $1 + 8 = 9$ nor $2 + 4 = 6$ are prime.
   - To get $a \cdot b = 27$ we must have $3 \cdot 9$ but $3 + 9 = 12$ is not prime.
   - To get $a \cdot b = 64$ we must have $8 \cdot 8$ but $8 + 8 = 16$ is not prime.

   The sum of all the numbers that work is $\boxed{11}$.

6. Since there are six children, X years ago his children's total age was $M - 6X$. Thus, $M - X = 2(M - 6X)$. Rearranging, this is $M - X = 2M - 12X$, or $11X = M$. Thus $M/X = boxed11$.

7. The shape described is shown below:



In the diagram there are three squares that are parallel to the edges of the paper, which are 10×10, 20×20 and 40×40. There are also two isosceles right triangles which legs of length 10 and 20. The total area of all these pieces is $100+400+1600+50+200 = \boxed{2350}$.

8. To be divisible by 72, the number has to be divisible by 8 and by 9. A number is divisible by 8 if and only if its last three digits are. That would require b to be 0, 2, 4, 6, or 8. Then to be divisible by 9, the sum of the digits must be divisible by 9. The sum of the digits of this number is $27 + a + b$. Trying all the possibilities for b, the corresponding values for a are 0 or 9, 7, 5, 3, and 1. That makes $a + 2b$ equal to 0, 9, 11, 13, 15, or 17. Squaring, the largest possible value is $17^2 = \boxed{289}$.

9. We will check through all the combinations of numbers that add to 10, and then find out how many ordered quadruples they make. For instance, $7 + 1 + 1 + 1 = 10$ can have four different orders, depending on which of a, b, c, or d is the 7. Here's a table:

Combination	$7+1+1+1$	$6+2+1+1$	$5+3+1+1$	$5+2+2+1$
Orders	4	12	12	12

$4+4+1+1$	$4+3+2+1$	$4+2+2+2$	$3+3+3+1$	$3+3+2+2$
6	24	4	4	6

The total is $4 + 12 + 12 + 12 + 6 + 24 + 4 + 4 + 6 = \boxed{84}$.

10. Multiplying a number that ends in 1 by itself any number of times results in a number that ends in 1. For numbers ending in 2, the final digit cycles 2, 4, 8, 6. Since the cycle is four numbers long, and $2025 \div 4$ has a remainder of 1, the 2025^{th} power will end in the first element of the cycle—2. Numbers that end in 3 work the same way, with a cycle of 3, 9, 7,1, so the 2025^{th} power will end in 3. And numbers that end in 4 have powers that alternate back and forth between 4 and 6. The 2025^{th} power will end in 4. So the units digit is the sum of these final digits: $1 + 2 + 3 + 4 = 10$ ends in $\boxed{0}$.

11. You could take the square root of the first equation and do a lot of messy arithmetic, or try this clever approach. Square out the first equation to get $x^2+2xy+y^2+2x+2y+1 = 54$. Multiply out the second equation to get $xy + x + y + 1 = 15$ so $xy + x + y = 14$. Double this to get $2xy + 2x + 2y = 28$. Now subtract from the first equation to leave $x^2 + y^2 + 1 = 26$. Subtracting 1 from both sides of this equation reveals that $x^2 + y^2 = boxed25$.

12. The problem implies that any number with at least two distinct digits leads to the same result (which is true!) so pick a number and get to work. We'll pick 1234:

$$\begin{aligned}4321 - 1234 &= 3087\\ 8730 - 0378 &= 8352\\ 8532 - 2358 &= 6174\\ 7641 - 1467 &= 6174\end{aligned}$$

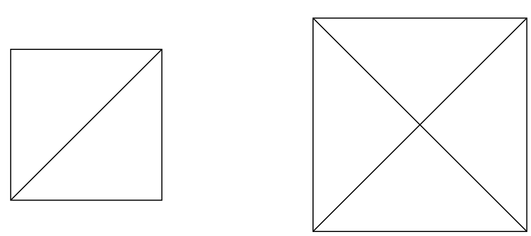
So $\boxed{6174}$ must be the the final destination of four digit numbers in this scheme. The number that are the final results of this operation for various numbers of digits are called *Kaprekar numbers*.

13. First, there are $\binom{8}{3} = 56$ ways to pick three of the eight dots. The $\binom{a}{b}$ notation is read "a choose b" and means to choose b objects from a selection of a objects, where order of choice is irrelevant. It can be found by determining how many ways there are to arrange the a objects in a line, where order does matter (which is $a \cdot (a-1) \cdot (a-2) \cdots 2 \cdot 1$ as there are a choices for the first object in the line, then $a - 1$ remaining choices for the second object, and so on). Then divide by the number of ways of rearranging the first b objects (the ones we select, because their order doesn't matter) and then dividing by the number of ways of rearranging the remaining $a - b$ objects (that we don't select, because their order no longer matters either). So $\binom{a}{b} = \frac{a!}{b!(a-b)!}$. In this case $\frac{8!}{3!5!} = 56$.

    Now among those 56 choices are 8 where all three dots lie in the same row (there are 2 rows, and in that row we could select all but any one of the 4 dots, and $2 \times 4 = 8$). Those three dots wouldn't form a triangle! So we subtract, leaving $\boxed{48}$ choices that do make triangles.

14. It's probably just easiest to start counting, ignoring primes and numbers that have more than two prime factors: 4, 6, 9, 10, 14, 15, 21, 22, 25, 26, 33, 34, 35, 38, 39, $\boxed{46}$.

15. Michael can combine two triangles into a square by placing their long sides together, or four by placing their short sides together:



Now these squares could be placed into 2×2, 3×3, etc. arrays to make larger squares. To make the most squares possible, Michael should use the smallest area squares he can until he runs out of triangles. So to make his squares will require 2, 4, 8, 16, 18, 32, 36, 50, 64, ... These are exactly the numbers that are either twice a square or four times a square. Since Michael only has 172 triangles, he must stop with the square that uses 50 triangles before he runs out of triangles. So he can make $\boxed{8}$ different sized squares.

16. There are $\binom{100}{2} = 4950$ pairs of points to choose from (see solution 13 for an explanation of the $\binom{a}{b}$ notation). But 100 of those points lead to edges of the figure, leaving $\boxed{4850}$ diagonals.

17. Since the rectangle has sides of 36 and 48, its diagonal has length 60, from the Pythagorean Theorem. Thus, the radius of the circle where the spheres intersect is half of this, or 30. Now the plane in which the intersection circle lies is equidistant from the centers of the two spheres, because the spheres have the same radius so the situation is symmetrical. Thus, from the center of the sphere to the center of the intersection circle is a distance of 16. Now since the corners of the rectangle lie on the intersection circle, they lie on the spheres themselves. The direction from the center of the sphere to the center of the intersection circle is perpendicular to the direction from the center of intersection to the points on the circle itself, including the corners of the rectangle. So we can use Pythagoras again to find that the distance from the center of the sphere to the point that lies on the sphere—the radius—is $\sqrt{30^2 + 16^2} = \boxed{34}$.

18. Since the angles of an equilateral triangle are $60°$ and the angles of a regular octagon are $135°$, the triangle rotates $165°$ around its vertex each time it rolls to a new side of the octagon. This is $\frac{165}{360} = \frac{11}{24}$ of a complete revolution. Now a circle of radius 5 has a circumference of 10π, so $\frac{11}{24}$ of this is a distance of $\frac{55\pi}{12}$. This is how far the vertex P moves as the triangle makes its first roll.

    But on the second roll, the triangle is rotating around vertex P, so it doesn't move at all! On the third roll, the same argument applies as on the first roll, and now the vertex P is not touching the octagon again. The next three rolls repeat this patter. We would repeat the pattern again, but the triangle only needs the first two of those three rolls to return to its original position (although P is now touching the octagon). So altogether, the vertex P moved five times and was stationary three times. During those five rolls, P moved a total distance of $5 \cdot \frac{55\pi}{12} = \boxed{275\pi/12}$.

19. Basically, the rows in the checkerboard, have a total of at most 8 checkers, and are arranged from top-to-bottom in decreasing order by length. So we want to find combinations of numbers that add up to 1, 2, ..., 8, but this time (as opposed to problem 9) the order is not important. So we will just count them. There is only one way to put one check on the board, two ways for two checkers (both in top row, or one in each of first two rows), and three ways for three checkers (3, $2 + 1$, $1 + 1 + 1$). There are five ways to put four checkers on the board (4, $3 + 1$, $2 + 2$, $2 + 1 + 1$, and $1 + 1 + 1 + 1$. For 5, 6, 7, or 8 checkers, the number of ways is 7, 11, 15, and 22 ways respectively. (There is no nice formula for this—you just have to count them all out!). The total is thus $\boxed{66}$.

20. Sneaky trick: let $x = \dfrac{5}{5+\frac{5}{5+\frac{5}{5+\cdots}}}$. Then $x = \dfrac{5}{5+x}$. Multiplying both sides by $x+5$ gives $x^2+5x=5$, so $x^2+5x-5=0$. Using the quadratic formula on this equation yields $x = \dfrac{-5\pm\sqrt{45}}{2}$. Now x must be positive, so we ignore the negative sign and simplify to find $x = \boxed{\dfrac{-5+3\sqrt{5}}{2}}$.