8^{th} Grade Individual Contest Solutions

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1. By trial and error, or by using a test for divisibility by 3, it is quickly found that 3 is a divisor of 561. Now $561 \div 3 = 187$. This can pretty quickly be factored as 11×17. So the divisors of 561 are 1, 3, 11, 17, $3 \times 11 = 33$, $3 \times 17 = 51$, 187, and 561. 561 has 8 factors, three of which are prime (note: 1 is *not* prime!), so it has $\boxed{5}$ factors which are not prime.

2. Since average is equal to the total score on all tests divided by the number of tests, Karthik currently has a total score of $5 \times 89 = 445$. To have an average of 90 on 6 tests, the total score would have to be $6 \times 90 = 540$. Thus, Karthik need $540 - 445 = \boxed{95}$ on his last exam.

3. To test if a number is divisible by 11, start with the units digit, then subtract the tens digit, then add the hundreds, subtract the thousands, and so on alternating adding and subtracting. If the final total is divisible by eleven, then the original number is divisible by eleven. (Compare this to the process of casting out nines, where all the digits are added to determine divisibility by nine.) That means that the two blanks have to add to either 3 or 14 for the number to be divisible by 11. So the replacements can be (from left-to-right) 1 and 2, 2 and 1, 3 and 0, 5 and 9, 6 and 8, 7 and 7, 8 and 6, or 9 and 5. That is a total of $\boxed{8}$ combinations.

4. Start counting up. 4829 has a 9, which is not a power of 2. 4830 through 4839 all have a 3, which cause them to fail. 4840 has a zero, which doesn't work. But $\boxed{4841}$ does work.

5. The smaller power of 2 can be factored out, leaving $2^{2022}(2^3 - 1) = 7 \cdot 2^{2022}$ so $m = 7$ and $n = 2022$, and their total is $\boxed{2029}$.

6. After n minutes, alarm a goes off if a is a divisor of n. So the problem is to find the smallest number n that has at least 16 divisors. Let n be factored into primes as $p_1^{r_1} p_2^{r_2} p_3^{r_3} p_4^{r_4}$ (we'll see in a moment why we don't need more than four primes). Then n has $(r_1+1)(r_2+1)(r_3+1)(r_4+1)$ factors. So one way of achieving 16 divisors is $r_1 = r_2 = r_3 = r_4 = 1$, and indeed after $2 \cdot 3 \cdot 5 \cdot 7 = 210$ minutes exactly 16 alarms will go off. Can we do better? Set $r_1 = 3$, $r_2 = r_3 = 1$ and $r_4 = 0$ gives $(3+1)(1+1)(1+1) = 16$ divisors, so another time at which 16 alarms go off would be $2^3 \cdot 3 \cdot 5 = 240$ but that is larger. Next, try $r_1 = r_2 = 2$, $r_3 = 1$, $r_4 = 0$ which sets off $(2+1)(2+1)(1+1) = 18$ alarms. And $2^2 \cdot 3^2 \cdot 5 = 180$ is smaller than 210. One more combination to try: $r_1 = r_2 = 3$, $r_3 = r_4 = 0$ gives $(3+1)(3+1) = 16$ divisors. But $2^3 \cdot 3^3 = 208$ is larger than 180, so $\boxed{180}$ is the winner.

7. First, notice that the 11-pound weight is actually irrelevant, since it can be made with a 3- and 8-pound weight combined. Now once you have a weight, you can add any multiple of 3 to

it. So Michael can obtain the weights 3, 6, 8, 9, 11, 12, 14, 15, 16, ... Since he just got three numbers in a row, all further weights can be obtained by adding more 3-pound weights. So the largest weight that cannot be obtained is $\boxed{13}$. Note that this is a case of the "Chicken McNugget Theorem" which says that with two weights a and b with no common factor, the largest unobtainable weight is $ab - a - b$. In this case it would be $3 \cdot 8 - 3 - 8 = 13$, as advertised.

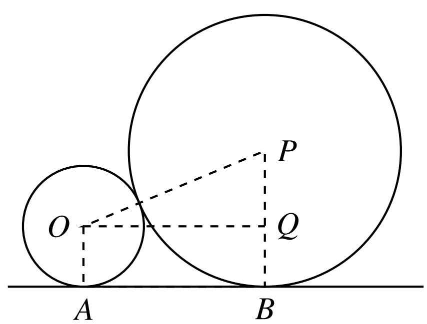
8. There is a formula for this in general. If the point is (X, Y) and the line is $Ax + By + C = 0$, then the distance is $\dfrac{|AX + BY + C|}{\sqrt{A^2 + B^2}}$. In this case, the result is $\dfrac{5 \cdot 6 + 12 \cdot 9 + 8}{\sqrt{5^2 + 12^2}} = \boxed{\dfrac{146}{13}}$. Now if you don't know this formula you can figure it out from scratch. To be perpendicular to the given line, a line would have to have slope $12/5$ (since the original line has slope $-5/12$ and perpendicular lines have opposite reciprocal slopes). So the equation of this line would be $y = \frac{12}{5}x - \frac{27}{5}$. Solve the two linear equations to determine the point of intersection of the two lines, and then find the distance of this point from the original point.

9. Notice that $2024! = 1 \cdot 2 \cdots 2023 \cdot 2024 = 2023! \cdot 2024$. So $2023!$ can be factored out of the numerator, and then cancelled with the denominator. Thus

$$\sqrt{\frac{2023! + 2024!}{2023!}} = \sqrt{\frac{2023!(1 + 2024)}{2023!}} = \sqrt{2025} = 45.$$

Now 45 has 3 and 5 as its prime factors: $\boxed{2}$ total prime factors.

10. Let the centers of the circles be O and P (O the center of the smaller circle) and the points of tangency A and B. Draw segments $\overline{OA}$, $\overline{PB}$, and $\overline{OP}$, together with $\overline{OQ}$ which is parallel to $\overline{AB}$ with Q lying on $\overline{PB}$.



Now notice that $\triangle OPQ$ is a right triangle, and by the conditions of the problem both OP and OQ are integers, while $PQ = 5$. That means that the Pythagorean theorem applies, and the only integer-sided right triangle with one leg of length 5 is the 5-12-13 triangle. Therefore, $OP = 13$ and this is the sum of the radii. The difference of the radii is 5, so $R^2 - r^2 = (R + r)(R - r) = 13 \cdot 5 = 65$ (R is the large radius, r is the small radius). Multiplying this last equation by *pi* shows that the difference in the areas of the circles is $\boxed{65\pi}$.

11. Since $\lfloor x \rfloor$ is always a whole number, let its value be n. Then $\frac{4}{5}x + \frac{1}{2} = n$. Solving for x: $x = \frac{5}{4}n - \frac{5}{8}$. Now we simply try various n to see which ones work. For example, when $n = 0$, x

would have to be $-5/8$, but $\lfloor -5/8 \rfloor = -1$, not 0. Trying $n = 1$ leads to $x = 5/8$ and $\lfloor 5/8 \rfloor = 0$, not 1. Similarly $n = 2$ doesn't work. For $n = 3$, x would be $25/8$ and $\lfloor 25/8 \rfloor = 3$ so this does work. Similarly, $n = 4, 5, 6$ all work, but 7 and beyond do not. So there are $\boxed{4}$ solutions.

12. Will's zucchini is 5% not water, which is 0.1 pound. After evaporation, the non-water part is 20% of the zucchini, or 1/5 of its weight. So the total weight is $\boxed{0.5}$ pounds.

13. Examine each slip to see if it needs to be turned over. The A slip does, because A is a vowel and we need to know if there is a prime on the other side of the paper. But the B slip does not; since B is not a vowel, it doesn't matter what is on the other side. E is a vowel and needs to be turned over. 3 does not need to be turned over, because whatever is on the other side, the statement in the problem is true. Similarly 5 does not need to be turned over. But 4 does, because if there is a vowel on the other side the statement would be false. So altogether $\boxed{3}$ of the slips must be turned over.

14. Of Roy's 125 initial small cubes, $3^3 = 27$ of them end up completely inside the large cube, and are not painted at all. 8 of them end up in the corners of the cube, and have three sides painted. The middles of the six faces of the large cube are only painted on one side, so that's $6 \cdot 3^2 = 54$ more cubes. That leaves $125 - 27 - 8 - 54 = \boxed{36}$ small cubes that lie along the edges of the large cube, and are painted on two sides. (Alternately, the large cube has 12 edges, and the middle three cubes in each edge are painted on two sides: $12 \times 3 = 36$.)

15. The corners of the quadrilateral are at $(0, 0)$, $(0, 6)$, $(2, 6)$, and $(5, 0)$. If we didn't cut off the shape at $y = 6$ and instead rotated the triangle with vertices $(0, 0)$, $(0, 10)$, $(5, 0)$ the result would be a cone of height 10 and base radius 5. The volume of a cone is $\frac{1}{3}\pi r^2 h$ so this cone has volume $250\pi/3$. By cutting off the triangle at $y = 6$ we cut off the top part of the cone, which has height $10 - 6 = 4$ and radius 2, giving the top part a volume of $16\pi/3$. So we subtract this from the full cone to obtain $234\pi/3$, meaning $a = \boxed{234}$.

16. Drop the altitude from A to $\overline{BC}$. By symmetry this is also a bisector of $\overline{BC}$. From the Pythagorean Theorem, then, this altitude has length 8, and $\triangle ABC$ has area $\frac{1}{2} \cdot 12 \cdot 8 = 48$. Now $BD = 2BA$ and $BE = 2BC$, so $\triangle BDE$ is similar to $\triangle ABC$ with sides twice as long, making its area four times as large, which is 192. To obtain the desired quadrilateral $ACED$ simply cut the original triangle ABC out of $\triangle BDE$, leaving an area of $192 - 48 = \boxed{144}$.

17. The original tetrahedron had six edges. At each of the four vertices, three more edges have been added, coming to a total of $6 + 4 \cdot 3 = \boxed{18}$.

18. Let's work this problem backward. For a number to have a digisquare of 1 requires it to be 1, 10, 100, ... How could we get a digisquare of 10? 31 and 13 work, as do 103, 130, 301, and 310. But no two-digit number could have a digisquare larger than $9^2 + 9^2 = 162$. Are any of those the digisquares of other numbers? Yes! The digisquare of 23 is 13, as is that of 32. No number has a digisquare of 31 or 103 as can be found by trial and error, but $130 = 9^2 + 7^2$, so both 79 and 97 have digisquares of 130, which then goes to 10, then to 1. A quick check shows that neither 98 nor 99 end up at 1 (both end up going in a cycle from 42 to 20 to 4 to 16 to 37 to 58 to 89 to 145 back to 42). So the answer is $\boxed{97}$.

19. The three vertices would form a right triangle if, and only if, two of them are opposite each other. This is because the triangle formed by the vertices is inscribed in the circumscribing circle of the octagon, and inscribed angles in a circle are half of the arc subtended by their endpoints, so for a 90 degree angle the endpoints would have to subtend a 180 degree arc—and be at opposite ends of a diameter.

    Now to count the possibilities, pretend that the vertices chosen have been painted red, green, and blue. Rotate the octagon so that the red vertex is at the top. There are now seven choices for the location of the blue vertex, leaving six choices for the location of the green, for a total of 42 combinations. Of these, there are six choices where the blue vertex is opposite the red (and the green can be in any of the six other spots), six more where the green is opposite the red, and six where the green and blue are opposite each other. Thus, 18 of the 42 possibilities lead to right angles, so the probability is $18/42 = \boxed{3/7}$.

20. After two rolls there are three possibiltiies: Michael won (which happens with probability $\frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$, Max won (probability $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$ or each player won one round, which happens the remainder of the time. Since if the game remains tied they continue to play we can ignore that eventuality. So the only thing we need to count is the $4/9 + 1/9 = 5/9$ of the time that someone wins. Thus, Michael wins $\frac{4/9}{5/9} = \boxed{\frac{4}{5}}$ of the time.