7^{th} Grade Individual Contest Solutions

IMSA *Mu Alpha Theta*

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1. Remember the order of operations: PEMDAS ("please excuse my dear aunt Sally") which stands for parentheses, exponents, multiplication, division, addition, subtraction. Multiple operations in the same expression are computed in that order, left to right. So:

$$\begin{aligned} 3 \cdot (1+5) - 7 \cdot 2 + 9 &= 3 \cdot 6 - 7 \cdot 2 + 9 \\ &= 18 - 7 \cdot 2 + 9 \\ &= 18 - 14 + 9 \\ &= 4 + 3 \\ &= \boxed{13}. \end{aligned}$$

2. Remember that "percent" literally means "divide by 100." So $\frac{22}{100} \cdot 300 = 22 \cdot 3 = \boxed{66}$.

3. Rectangles have two pairs of equal sides. So two of the sides have length 5 so to make a total length of 22 the remaining two sides must have length 6 ($\frac{22-2\cdot 5}{2} = \frac{12}{2} = 6$). With sides of 5 and 6, the area is $5 \cdot 6 = \boxed{30}$.

4. Adding, $1 + 3 + 5 + \cdots + 19 = 100$, and $\sqrt{100} = \boxed{10}$. Alternately you can note the patters: $1 = 1$, $1 + 3 = 4$, $1 + 3 + 5 = 9$, $1 + 3 + 5 + 7 = 16$, and in general the sum of the first n odd numbers is n^2. Taking the square root gives you n.

5. You can do the arithmetic, squaring 95 and subtracting 100 and dividing by 105. Or you can factor: $\frac{n^2-100}{n+10} = \frac{(n-10)(n+10)}{n+10} = n - 10$, so substituting $n = 95$ yields $95 - 10 = \boxed{85}$.

6. Ideally, to get the largest you should take the largest numbers. However, if we tried to multiply -18, 13, and 16 the result would be negative, making it the smallest possible product, not the largest! So the next number in size is -12. Replacing the 13 with it gives $-18 \cdot -12 \cdot 16 = \boxed{3456}$.

7. For numbers that are at least n^2 but less than $(n+1)^2$, their square root is at least n but smaller than $n+1$, so the greatest integer is not larger than this value is n. So $\lfloor \sqrt{n} \rfloor$ is 1 for n from 1 to 3, 2 for n from 4 to 8, 3 for n from 9 to 15, and so forth. So the total is $3 \cdot 1 + 5 \cdot 2 + 7 \cdot 3 + 9 \cdot 4 + 11 \cdot 5 + 6$ (the last term coming from the $\lfloor \sqrt{36} \rfloor$). Adding, $3 + 10 + 21 + 36 + 55 + 6 = \boxed{131}$.

8. Just like you can add up the digits of a number to determine if it is divisible by 9 (a process called "casting out nines"), you can cast out elevens, but instead of adding the digits you alternately add-then-subtract as you read from right to left. In this case, $1 - X + 4 - 2 + 9 - 5 = 7 - X$. To be divisible by eleven, this digital total must be zero. So $7 - X = 0$ and $X = \boxed{7}$.

9. Let's enumerate the number of grains of rice on each plate for a while: 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ... Note that after the initial 1, the units digits repeats the pattern 2, 4, 8, 6 forever. So between plates 2 and 29 (inclusive) the units digit will total to zero. That leaves 1 + 2 + 4 + 8 for the grain on plates 1, 30, 31, and 32. Since this totals to 15, the units digit is $\boxed{5}$.

10. Think about "unwinding" the vine while always keeping it at the same angle with the horizontal. This will result in the vine being stretched out away from the pole at a 30° angle to the ground reaching a height of 10 feet. Then, together with the pole, the vine would form half of an equilateral triangle. Since the vine is a full side of this triangle and the 10-foot pole is only half the side, a full side–the length of the vine—is $\boxed{20}$ feet.

11. Since Max doesn't have twice as many pennies as Will does, doubling Will's pile will give him more pennies than Max. Giving Max only one new penny (while also giving Will one) doesn't put him any farther ahead, so that if Will doubles his pile he will have more than Max. Thus, it seems the proper start is to double Max's pile, while giving Will just one penny for a while, then giving them each one penny per round until Will is just under half of Max's total, then doubling Will while giving Max one last penny. The best strategy is as follows:

Round	Action	Will	Max
0		7	9
1	Will +1, Max dbl	8	18
2	Will +1, Max +1	9	19
3	Will +1, Max +1	10	20
4	Will +1, Max +1	11	21
5	Will dbl, Max + 1	22	22

So it takes $\boxed{5}$ rounds for the players to achieve equality.

12. First, note that the 11-pound weight is really immaterial to the problem, since it can easily be made from one 3- and one 8-pound weight. The easiest way to solve this problem is by trial and error. Since there is a gap between the 8-pound weight and the 11-pounds we can make, we note that there is no way to make a 10-pound weight (a 9-pounder can be made from three 3-pound weights). Is this largest? The next ones to check are adding 3- and 8-pound weights to this impossible weight. Can a 13-pound weight be made? No, but the 18-pound can (six 3-pound weights). Moving up from there, $14 = 8 + 3 + 3$ and $15 = 3 + 3 + 3 + 3 + 3$ while $16 = 8 + 8$. So three weights in a row can be made up, and then by adding extra 3-pound weights to these combinations every weight from there on up can be made. So $\boxed{13}$ is the largest weight that cannot be made.

    This is a special case of a general problem. If you have a- and b-pound weights, and a and b share no common factors, then the largest weight that cannot be made is $(a - 1)(b - 1) - 1$. Note that $(3 - 1)(8 - 1) - 1 = 13$, the answer to the original problem. This is usually called the "Chicken McNugget Theorem" because sometimes it's asked about the possible number of McNuggets that can be ordered. It's more official name is the Frobenius Coin Theorem, named after a mathematician who studied generalizations of this problem.

13. Notice that among the other numbers, there are two that can be added to 3 to obtain a prime: $3 + 4 = 7$ and $3 + 8 = 11$. 4 also has two "companions:" 3 and 13. 5 and 8 also have two companions each. But 12 and 13 only have *one* number in the list that they can be added to in order to obtain a prime. Therefore, those two numbers must go on the ends. The two possible lines are 12, 5, 8, 3, 4, 13, and its reverse 13, 4, 3, 8, 5, 12. In either case, the first and last numbers add to $\boxed{25}$.

14. Might as well just start testing:

$$84 \to 8^2 - 4^2 = 64 - 16 = 48 \text{ which cycles back to itself.}$$
$$83 \to 8^2 - 3^2 = 64 - 9 = 55 \to 5^2 - 5^2 = 0.$$

That didn't take long! Quick trial and error shows that $\boxed{83}$ is sneaky.

15. Let's say you factor a number n into primes: $n = p^a q^b r^c \ldots$ where p, q, r and so forth are primes. So, for example, $600 = 2^3 \cdot 3^1 \cdot 5^2$. Then the number of factors of n is $(a + 1)(b + 1)(c + 1) \ldots$. This is because you could choose anywhere from zero to a factors of p, zero to b factors of q, and so forth.

Now the fifth alpha number is $12 = 2^2 \cdot 3$ with $(2 + 1)(1 + 1) = 6$ factors. To have 7 factors, you would need a sixth power, so the smallest contestant is $2^6 = 64$. But $8 = 4 \cdot 2 = (3 + 1)(1 + 1)$ factors can be found with $2^3 \cdot 3^1 = 24$, so 24 is the sixth alpha number. The seventh alpha number must then have at least $9 = (2 + 1)(2 + 1)$ factors, which is achieved by $2^2 \cdot 3^2 = 36$. Ten factors can be achieved by $10 = (4 + 1)(1 + 1)$ which is the number of factors $2^4 \cdot 3^1 = \boxed{(}48)$.

16. Let's find the factorials of the digits first. $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, $6! = 720$, $7! = 5040$ has more than three digits, so none of the digits of our number can be 7, 8, or 9.None can be 6 either, because the sum of the factorials would then be at least 722, so the sum would have a 7, 8 or 9, or be larger than three digits, which are not allowed.

On the other hand, there must be at least one 5, because $4! + 4! + 4! = 72$ is not a three-digit number. But even with three 5's, the sum of the factorials of the digits would only be 360, so the hundreds digit cannot be the 5, and will be a 1, 2, or 3. There cannot be two fives, because the hundreds digit is not five, and none of the possibilities 155, 255, or 355 work.

Since there is exactly one 5, and the hundreds digits is at most 3, the next largest possibilities are 345 or 354. Neither of these are factorions, and more importantly, the sums of the factorials of their digits are 150. Thus, the hundreds digit of the number must be a 1 and the number must be at most 150. The tens digit must be 2, 3, 4, or 5. We are now down to few enough possibilities to consider them all: 125, 135, 145, and 150. Of these, only $\boxed{145}$ checks.

17. Notice that $4-1 = 3$, $13-4 = 9$, $40-13 = 27$, and $121-40 = 81$. The numbers in this sequence are increasing by powers of 3. So the next number should be $121 + 3^5 = 121 + 243 = 364$ and $364 + 3^6 = 364 + 729 = 1093$. Adding these two gives $1094 + 364 = \boxed{1458}$.

18. If S is the magic sum, then the numbers in each vertical column add up to S. Since there are nine vertical columns, the numbers in all nine columns add up to $9S$. But the numbers in all nine columns are all the numbers in the magic cube, so also add up to $1+2+3+\cdots+27 = 378$. So the magic sum must be $S = 378/9 = \boxed{42}$.

19. Factor $x^2+13x = x(x+13)$. The numbers x and $x+13$ can never have any common factors other than 13, but if x is a multiple of 13, say $x = 13y$, then $x(x+13) = 13y(13y+13) = 13^2y(y+1)$ which would mean that both y and $y+1$ are squares, forcing $y = 0$ so that x would not be positive. This case is ruled out.

    So since there are no common factors, both x and $x+13$ must be squares. The two positive squares that are 13 apart from each other are 36 and 49. So $x = \boxed{36}$.

20. There are $2^5 = 32$ possible combinations of heads and tails. The scale can only go out of balance after an even number of flips, because after an odd number of flips there is no way for the scale to have exactly two more marbles on one side than on the other (the total number of marbles must be even after an even number of flips). So there are two ways (HT and TH) for the first two flips to occur and not have the scale go out of balance. If one of these possibilities occurs, the scale is in perfect balance after two flips, and there are again two ways for the next two flips to keep it from going out of balance. Finally, if the scale is still in balance after four flips, there are two ways for the fifth flip to occur (H or T), and once again the scale will not go out of balance. Thus, there are $2 \times 2 \times 2 = 8$ possible combinations of flips for which the scale stays in balance. The probability is then $8/32 = \boxed{1/4}$.