8^{th} Grade Team Contest

IMSA *Mu Alpha Theta*

March 10, 2021

1. Find the six-digit number $abcdef$ that satisfies the following requirements: a is divisible by 2. ab is dividible by 3, abc is divisible by 4, d is divisible by 5, de is divisible by 6, def is divisible by 7. The digits a, b, c,d, e, and f are all different, and they add to 20.

2. Find the number of distinct primes that divide 10!. (Remember that $n! = 1 \cdot 2 \cdot 3 \cdots n$.)

3. Find the largest prime whose square is a divisor of 2021!.

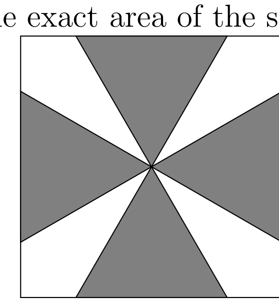
4. Ken's clock is set to the wrong time. He notes that he could adjust the clock to the correct time by rotating both hands 60 degrees clockwise and then reflecting the hands in the vertical axis. If the correct time is 10:40, what is the incorrect time currently showing on the clock?

5. The number a is a positive integer, and the average of all positive integers between a and $4a$ (inclusive) is 75. Compute a

6. In music theory, an *octave* contains 12 *semitones*, which are commonly called "notes," arranged in increasing order. Two notes different notes are chosen within the same octave. What is the probability that these two notes are exactly 7 semitones apart?

7. There are two 3-digit number whose digits are in descending order and the product of whose digits is 126. Find their sum.

8. The figure below shows four congruent equilateral triangles inside a square. If the area of each triangle is 2, compute the exact area of the square.



9. Given that the average of three positive integers is 5, determine the maximum possible value for the sum of their squares.

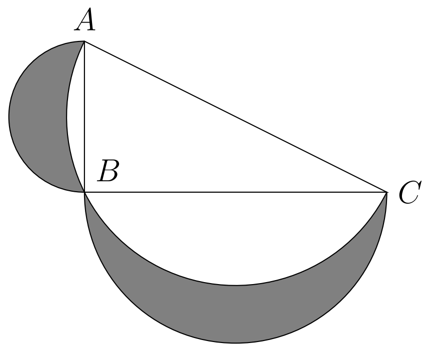
10. In the country of Sixylvania, the money works like this: there are 6 pennies to the sixel, 6-sixels to the sixo, and 6 sixos to the sixar. In the neighboring country of Icosania they use the same pennies, but there are 4 pennies to the quarkel, 5 quarkels to the ico, and 12 icos to the icar. What is the minimum number of sixars that can be exchanged for a positive whole number of icars?

11. How many four-digit numbers are there that contain a 2 or a 3 (or both) as a digit?

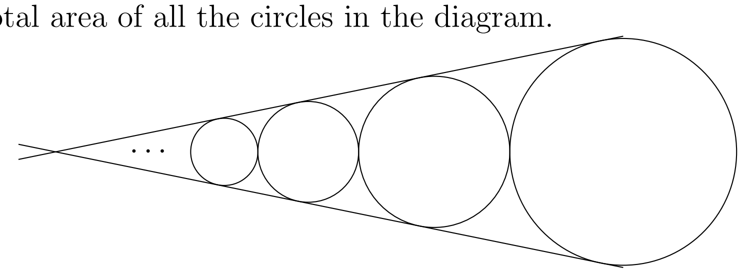
12. Determine the probability that a randomly chosen divisor of 999,999 is divisible by 3.

13. Consider all six-digit palindromes. The positive integer $x > 1$ is the most common divisor of these palindromes. That is more of these palindromes are divisible by x than any other positive integer. What is x?

14. In the figure below, $\triangle ABC$ is a right triangle with right angle at B and an area of 10 square units. Semicircles are drawn with the three sides of the triangle as their diameters, overlapping as shown. The shaded area is inside the semicircles drawn with diameters $\overline{AB}$ and $\overline{BC}$ but outside the semicircle with diameter $\overline{AC}$. Compute the total area of the shaded region.



15. In the diagram below, the largest circle has a radius of 6, and it is tangent to a circle of radius 4. The two common external tangent lines are drawn, and then more circles are added working leftward, each new circle tangent to the previous circle and to both lines. Compute the total area of all the circles in the diagram.



16. Aggie's bad contains 4 white marbles and 3 black marbles. Ollie's contains 2 white and 4 black marbles. Each takes two marbles from their bag at random. What is the probability that both took out the same marbles? (That is, find the probability that both removed two white marbles, or the both removed two black marbles, or that both removed one of each, combined.)

17. Two real numbers a and b satisfy the equations $a^2+b^2 = 28$ and $a^4+b^4 = 1559/2$. Given that $a+b$ is a positive integer, find it.

18. You have a number line in front of you and a number of colored tiles. Red tiles are one unit long, blue tiles are two units long, and yellow tiles are three units long. In how many different ways could you completely cover the number line from 0 to 10 with tiles (no overlapping tiles, no gaps between tiles)?

19. Ahsoka has a supply of chips where each chip is labeled with a number from "1" to "5" inclusive. Ahsoka takes these chips and places them in bags. Each bag has exactly three chips, and there is exactly one bag with each possible combination of chips in it. Thus, there is a bag with chips 1-2-3, one with 2-2-4, and even one with 5-5-5. But 3-2-1 counts as the same combination as 1-2-3, so there is not a separate bag for that combination. After making all these bags, Ahsoka picks a bag at random. What is the probability that all three chips in the bag have the same label?

20. You are playing a game where you have two tokens and your opponent has four tokens. At each turn you flip a coin, and if it is heads you have to give your opponent a token, but tails forced your opponent to give you a token. The first player to collect all the tokens wins. What is your probability of winning?